

SHORT LOOPS IN SURFACES WITH A CIRCLE BOUNDARY COMPONENT

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ABSTRACT. It is a classical theorem of Loewner that the systole of a Riemannian torus can be bounded in terms of its area. We answer a question of a similar flavor of Robert Young showing that if T is a Riemannian 2-torus with boundary in \mathbb{R}^n , such that the boundary curve is a standard unit circle, then the length of the shortest non-contractible loop in T is bounded in terms of the area of T .

1. INTRODUCTION

Robert Young in [10] conjectures the following: There is a constant $M > 0$ such that if $K \subset \mathbb{R}^n$ is an embedded torus with one boundary component and ∂K is a unit circle, then there is a closed curve of length ℓ in K which is not null-homotopic and satisfies $\ell^2 \leq M(\text{area } K - \pi)$.

Note that the area of a disk bounding the unit circle is π so by a surgery one can get a torus with boundary K with area arbitrarily close to π . What the conjecture really says is that if the area is close to π then necessarily there is a ‘short’ non null homotopic curve in K . Clearly if the area is much bigger than π , say 2π , then the conjecture follows from the classical result of Loewner [9].

The purpose of this note is to show that this conjecture holds. In fact we show a slightly stronger result namely that the inequality holds for the length of a non-separating simple closed curve in K . We show further that this result also holds for any orientable surface S with a single boundary component equal to the unit circle.

I would like to thank S. Sabourau for suggesting that my proof applies to higher genus surfaces as well.

2. LENGTH AREA INEQUALITY AND SYSTOLES

We will use the co-area formula [4, Theorem 13.4.2], which we state now in a simplified form:

Length area inequality. *Let M be a Riemannian 2-manifold and let $f : M \rightarrow \mathbb{R}$ be a 1-Lipschitz function. Then*

$$\text{area}(M) \geq \int \text{length}(f^{-1}(t)) dt.$$

We recall also Loewner's inequality

Loewner's inequality. *Let T be a Riemannian 2-torus. Then T has a non-contractible geodesic γ of length ℓ satisfying $\ell^2 \leq \frac{2}{\sqrt{3}} \text{area} T$.*

We state now our main result:

Theorem 2.1. *Let T be a Riemannian torus with a single boundary component embedded in \mathbb{R}^n . Assume that ∂T is isometric to the unit circle. Then there is a non separating simple closed curve in T of length ℓ such that*

$$\ell^2 \leq 10^3(\text{area } T - \pi).$$

Proof. Let ℓ be the length of the shortest non separating simple closed curve in T .

Without loss of generality we may assume that ∂T lies on the xy -plane. We may further assume that the functions $X : (x_1, \dots, x_n) \rightarrow x_1$ and $Y : (x_1, \dots, x_n) \rightarrow x_2$ are Morse functions for T . Indeed a slight linear perturbation of X, Y gives Morse functions [5, p.43], and this slight perturbation won't affect significantly the calculations that follow. Alternatively this can be obtained by slightly deforming S .

We note that if $X^{-1}(t)$ contains a non separating loop then $\text{length}(X^{-1}(t)) \geq \ell$. Note also that by Morse Theory if for some $a < b$ $X^{-1}(a), X^{-1}(b)$ contain a non separating loop then $X^{-1}(t)$ contains a non separating loop for all $a < t < b$. We remark that for each $t \in [-1, 1]$, $X^{-1}(t)$ contains a simple arc α_t spanning $X^{-1}(t) \cap \partial T$. Clearly α_t has length greater than the corresponding geodesic γ_t joining the same points. Assume now that for some $a < b$ with $b - a > \ell/10$ $X^{-1}(a), X^{-1}(b)$ contain a non separating loop. Then by the length-area inequality

$$\text{area}(T) \geq \int \text{length}(\alpha_t) dt + \ell^2/10 \geq \pi + \ell^2/10.$$

It follows that in this case the theorem holds as $\ell^2 < 10^3(\ell^2/10)$.

Similarly if the set of t for which

$$\text{length}(\alpha_t) \geq \text{length}(\gamma_t) + \frac{\ell}{10}$$

has measure greater or equal to $\ell/100$ then

$$\text{area}(T) \geq \int \text{length}(\alpha_t) dt \geq \pi + \ell^2/1000$$

and the theorem holds.

Let $[a_1, b_1], [c_1, d_1]$ be maximal intervals with the property that $X^{-1}(a_1), X^{-1}(b_1)$ contain a non separating loop and $Y^{-1}(c_1), Y^{-1}(d_1)$ contain a non separating loop. Clearly $b_1 - a_1 \leq \ell/10, d_1 - c_1 \leq \ell/10$. For each $t \in [-1, 1]$ denote by β_t the simple arc in $Y^{-1}(t)$ spanning $Y^{-1}(t) \cap \partial T$ and by γ'_t the geodesic arc joining the same points.

By the previous argument there are a, b, c, d with $0 \leq a_1 - a \leq \ell/100, 0 \leq b - b_1 \leq \ell/100, 0 \leq c_1 - c \leq \ell/100, 0 \leq d - d_1 \leq \ell/100$ such that for $z \in \{a, b\}$

$$\text{length}(\alpha_z) - \text{length}(\gamma_z) \leq \ell/10$$

and for $z \in \{c, d\}$

$$\text{length}(\beta_z) - \text{length}(\gamma'_z) \leq \ell/10.$$

We consider now the union of arcs:

α_a, α_b restricted to $[c, d]$ and β_c, β_d restricted to $[a, b]$. This union is a separating simple closed curve w on T . Let's denote by T_1 the connected component of $T \setminus w$ containing ∂T and by T_2 the other connected component of $T \setminus w$. We remark that w has the following properties:

1. $\text{length}(w) \leq \frac{8\ell}{10} + \frac{8\ell}{100} < \frac{9\ell}{10}$.
2. The shortest non separating loop in T is homotopic to a loop contained in T_2 . Indeed every loop in T_1 is separating and any arc with endpoints on w is homotopic to a subarc of w .

We note that already property 1 above suffices to answer Young's original question if we interpret ℓ in the previous part of the proof to be the length of the shortest non-null homotopic loop.

By the isoperimetric inequality

$$\text{area}(T_1) \geq \pi - \frac{1}{4\pi} \left(\frac{9\ell}{10} \right)^2 \quad (*).$$

We fill w by a disk D of arbitrarily small area to obtain a torus $T' = D \cup T_2$ of area less or equal to $\text{area}(T_2) + \epsilon$ for some arbitrarily small $\epsilon > 0$. By choosing carefully the metric on the gluing we can make sure that T' is a smooth riemannian manifold.

If ℓ_1 is the length of the shortest non separating loop on T' since ∂T_2 has length less than $9\ell/10$ we have that $\ell \leq \ell_1 + \frac{9\ell}{20}$ so $\ell \leq 2\ell_1$.

Applying Loewner's inequality to T' we have that

$$\ell_1^2 \leq \frac{2}{\sqrt{3}}(\text{area}(T_2) + \epsilon)$$

and since this holds for any $\epsilon > 0$ we obtain

$$\text{area}(T_2) \geq \frac{\sqrt{3}}{2}\ell_1^2 \Rightarrow \text{area}(T_2) > \frac{\ell^2}{8}.$$

Combining this with (*) we have

$$\text{area}(T) = \text{area}(T_1) + \text{area}(T_2) > \pi + \frac{1}{100}\ell^2.$$

This inequality clearly implies the theorem. \square

Theorem 2.2. *Let S be a Riemannian surface with a single boundary component embedded in \mathbb{R}^n . Assume that ∂S is isometric to the unit circle. Then there is a non separating loop in S of length ℓ such that*

$$\ell^2 \leq C(\text{area } S - \pi)$$

where C is a universal constant that does not depend on S .

Proof. The argument of the previous theorem applies with little change. Let ℓ be the length of the shortest non separating simple closed curve in S . If $X : (x_1, \dots, x_n) \rightarrow x_1$ is a Morse function for S (where ∂S lies on the xy -plane) then in this case there are $a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n$ such that all non-separating loops of S lie in $X^{-1}([a_1, b_1] \cup \dots \cup [a_n, b_n])$ where $\sum(b_i - a_i) \leq \ell/10$. Using in a similar way the Morse function $Y : (x_1, \dots, x_n) \rightarrow x_2$ and arguing as in the previous theorem we arrive at a collection of separating simple closed curves w_1, \dots, w_k each of which has length less than ℓ . If we denote by S_i the connected component of $S \setminus w_i$ that does not contain ∂S then we may apply Gromov's generalization of Loewner's inequality for S_i (see [7, sec. 2.C], [6, Cor. 5.2.B]) to obtain the desired bound on ℓ . \square

3. DISCUSSION

Theorems 2.1, 2.2 'quantify' the defect of a filling of S^1 by a general surface rather than a disc. They say that if the filling by a surface is 'close' to the optimal filling by a disc then the surface is 'close' to a disc as it has a short non-separating geodesic. This has a similar flavor to the classical Bonnesen inequality [3] on isoperimetric defect quantifying how far is a region from being optimal for the isoperimetric inequality. A strengthening of Loewner's inequality in this spirit is given in [8].

We note also that Babenko in [1] has studied systoles of manifolds with boundary.

A related well known question to theorems 2.1, 2.2 is the conjecture of Gromov on the filling volume (area) of S^1 [6, sec.5.5, p.60]. One wonders if the analog of theorem 2.1 holds in this case, namely whether if T is a torus with boundary filling S^1 then there is a non-separating curve of length ℓ in T satisfying

$$\ell^2 \leq C(\text{area } T - 2\pi)$$

for some universal constant C . Note that the Gromov's conjecture is known to hold for tori with boundary [2] but is still open for higher genus surfaces.

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